Journal of Engineering Mathematics **37:** 191–209, 2000.

Orthotropic viscous response of polar ice

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Received 22 May 1998; accepted in revised form 6 May 1999.

Abstract. Re-orientation of individual crystal glide planes as isotropic surface ice is deformed during its passage to depth in an ice sheet creates a fabric and associated anisotropy. A simple macroscopic description is that these material glide planes are rotated towards planes normal to an axis of compression, and away from planes normal to an axis of extension, inducing an instantaneous orthotropic viscous response with reflexional symmetries in the planes orthogonal to the current principal stretch axes. An associated orthotropic viscous law expresses the stress in terms of the strain-rate, strain, and three structure tensors based on the principal stretch axes. The fabric induced during differential stretchings along fixed principal axes, and the subsequent instantaneous viscous shear response in different planes due to the frozen fabric when the axial stress and strain-rate are removed, define a set of instantaneous directional viscosities in terms of the frozen principal stretches and the material response coefficients. Various inequalities and equalities between these viscosities are derived from the original rotation concepts, which, together with observed enhancement factors at large stretch and shearing, impose restrictions on the permitted response coefficients. It is shown how a simple viscous law can meet all these requirements, and such a law is illustrated for continued axial stretchings and shearing.

Key words: polar ice, induced anisotropy, orthotropy.

1. Introduction

Ice core samples taken from depth in an ice sheet reveal strong fabrics, shown by significant alignment of initially randomly distributed *c*-axes of individual crystals, and consequent substantial differences in shear viscosities in different planes. The conventional incompressible nonlinearly viscous fluid law used for ice sheet dynamics cannot reflect such induced anisotropy, nor can any simple fluid law, since such laws are necessarily isotropic by frame indifference. However, ice sheets do flow over long time scales, and a macroscopic constitutive law which describes an anisotropic viscous shear response which changes with the evolving fabric, in which shear strain-rate vanishes at zero shear stress, is an appropriate description. Further, the relations should evolve continuously from an isotropic viscous law in an initial state with no fabric, and should again become an isotropic viscous law if the evolving fabric becomes isotropic.

A basic, and physically motivated, approach is to construct a macroscopic law from the properties of an individual crystal and assumptions on how crystal interactions yield an average response. Azuma [1] and Azuma and Goto-Azuma [2] suppose that individual crystals deform only by basal glide, and its direction is determined by that of the maximum macroscopic shear stress in the polycrystal, and the crystal (microscopic) and polycrystal (macroscopic) stresses are related by a geometric tensor associated with the *c*-axis and glide directions. The microscopic shearing is assumed to satisfy a viscous power law (Weertman [3]), and an averaging procedure determines the macroscopic response. The model is used to predict fabric evolution (c-axis orientation changes) for various stress configurations, and numerical simulations are compared with field observations. Van der Veen and Whillans [4] adopt a similar approach, but make a different assumption that the macroscopic average stress acts on all individual crystals. They also include a recrystallisation process, and consider two alternative models. Numerical simulations illustrate c-axis orientation evolution for various stress loading, and particularly the influence of recrystallisation.

An alternative approach is the viscoplastic self-consistent theory based on Hutchinson's [5] treatment of the creep of polycrystalline materials and its extension by Molinari *et al.* [6]. Here the single crystal is treated as an embedded idealised geometric inclusion in an infinite medium with properties of an assumed form supposed to represent the macroscopic behaviour. The response to uniform loading at infinity for given crystal properties determines the medium properties. Castelnau *et al.* [7] consider crystal slip on basal, prismatic and pyramidal planes, and allow stress and strain-rate to depend on the crystallographic orientation. The self-consistent method determines the instantaneous anisotropic viscous response of the medium. Meyssonnier and Philip [8] apply this approach to a simplified configuration, namely for a transversely isotropic medium with the inclusion an ellipsoid geometrically aligned with the symmetry axes, which can apply only to loading and flow which reflects this symmetry. They also introduce an orientation-distribution function to measure weightings of a continuous spectrum of c-axis orientations. Simulations for uni-axial stress illustrate c-axis evolution and sensitivity to some of the model parameters.

For practical purposes, a constitutive law that is useful for investigating the large-scale dynamics of an ice sheet must be a relatively simple relation between stress and a limited number of variables representing the deformation and structure. It must also be a valid law, satisfying the principle of material frame indifference, which requires material properties to be independent of the observer. Svendsen and Hutter [9] formulate directly a frame-indifferent viscous law which incorporates fabric through a single structure tensor defined by an axis of assumed transverse isotropy. Again, an orientation distribution function is introduced to give continuous weighting to the axis orientation, and an evolution equation for the distribution function is investigated and illustrated for shearing deformation. Gödert and Hutter [10] have extended this theory. The complicated calculations required to follow the evolving properties of individual ice elements will add considerably to numerical treatments of large ice sheet flows. A transversely isotropic flow law that avoids the use of an orientation distribution function has been formulated by Van der Veen and Whillans [11]. They modify Johnson's [12] law for a transversely isotropic viscoelastic solid, based on a general constitutive law for a transversely isotropic medium derived by Ericksen and Rivlin [13], by replacing material measures of stress and strain-rate by spatial measures. However, they include the vertical (gravity) direction in the material structure, so it is not a valid constitutive relation for the response to general loading. The predictions of this flow law are illustrated by comparing results of numerical simulations with field measurements.

An alternative approach which requires only that the deformation gradient of each element is determined during the ice flow was adopted by Morland and Staroszczyk [14]. The macroscopic viscous law proposed was motivated by a simple picture of individual crystal glide planes, material planes, being rotated towards planes normal to principal axes of compression, and away from planes normal to principal axes of extension. Given that the initial isotropy implies a random distribution of crystal glide planes symmetrically distributed about all planes, it is supposed that the new orientation will then be distributed symmetrically about these principal stretch planes, and so the new instantaneous viscous response will have reflexional symmetry in these planes. That is, the instantaneous viscous response is orthotropic with respect to the current principal stretch planes whose normals are the principal stretch axes, so that the base planes of the orthotropy are evolving. The directional strengths of the response depend on the current deformation, and there must be dependence, at least, on differences between the principal stretches, which, according to the rotation picture, govern the rotations of glide planes towards and away from the principal stretch planes. This overview ignores the local interactions between individual crystals, and assumes that the macroscopic mechanical response can be described in terms of fabric induced purely by macroscopic deformation. It further supposes that the induced anisotropy depends only on the evolving current deformation, and does not depend on the deformation path. In practice, the effects of crystal interactions may depend on the nature of the deformation process, and therefore induce a different fabric for different deformation histories. However, this approximation is the most simple approach to an evolving anisotropic viscous constitutive law which will be tractable in a theory of large scale ice sheet dynamics. In this first exploration, the influence of temperature on fabric was not considered. This formulation of a viscous constitutive law is not restricted to instantaneous ‘snapshots’, nor to a fully developed fabric in which rotations of glide planes are no longer occurring.

The orthotropic viscous response was described in terms of three structure tensors defined by the outer products of the three orthogonal vectors along the principal stretch axes. The viscous law is then a frame-indifferent relation between stress, strain-rate, deformation and the three structure tensors, for which a general representation is available. Three different classes of law were considered, depending on the choice of stress, strain-rate, deformation, and structure tensors adopted. It was assumed that the deviatoric stress vanishes when the strain-rate vanishes, to give the fluid-like behaviour, and that the law reduces to an isotropic viscous fluid law in the initial state without deformation, and in any subsequent deformed state which has equal principal stretches, necessarily unity by the incompressibility assumption. For each class, only the same set of terms contribute to the instantaneous directional shear responses following differential stretchings along fixed principal axes, and a simple model with a single fabric response function was adopted to illustrate how some of the expected qualitative behaviour could be realised. This model, however, did not have the flexibility to allow different directional viscosities to be correlated with observed responses.

Here we adopt the Morland–Staroszczyk [14] theory and derive from the rotation concepts further inequalities and equalities which must be satisfied by the instantaneous directional viscosities following axial stretchings, depending on the three principal stretches. One reverses a postulated inequality in [14]. Adopting the form of orthotropic law expressed in terms of Cauchy stress and current strain-rate, and restricting attention to the terms contributing to these responses, we re-examine the corresponding viscosity relations derived in [14]. The relations are separable in the isotropic dependence on strain-rate and fabric dependence on deformation, and a simplified form has two fabric-response functions with dependence on the principal stretches and an invariant measure of total deformation. We show how one of the viscosity equalities relates the two functions, so that the response can again be described in terms of a single function, but now the required inequalities can be achieved by a simple function with one or more free parameters. The parameters change the detail of the maintained axial and shearing responses, and illustrations are presented for example functions. This theory, with a single function, allows good qualitative correlation with observed responses, and flexibility to correlate with more detailed experimental results.

2. Orthotropic viscous model

We now follow the theory proposed by Morland and Staroszczyk [14], but consider only one of the three proposed classes of orthotropic viscous law to demonstrate how the viscosity inequalities and enhancement factors can be realised. The chosen form is the relation between the frame-indifferent deviatoric Cauchy stress $\hat{\boldsymbol{\sigma}}$, current strain-rate $\mathbf{D}$, Cauchy–Green strain tensor $\mathbf{B}$ and three structure tensors $\mathbf{M}^{(r)}$ $(r = 1, 2, 3)$ defined by the outer products of the current principal-stretch-axes unit vectors $\mathbf{e}^{(r)}$ $(r = 1, 2, 3)$. The alternative classes were both relations between frame-invariant measures.

Let Ox_i $(i = 1, 2, 3)$ be spatial rectangular Cartesian co-ordinates with OX_i $(i = 1, 2, 3)$ particle reference co-ordinates, and v_i the velocity components, then the deformation gradient $\mathbf{F}$, spatial velocity gradient $\mathbf{L}$ and strain-rate $\mathbf{D}$ have components

$$F_{ij} = \frac{\partial x_i}{\partial X_j}, \qquad L_{ij} = \frac{\partial v_i}{\partial x_j}, \qquad D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{1}$$

and the deformation gradient is determined by the kinematic relation

$$\dot{F}_{ij} = \frac{\partial F_{ij}}{\partial t} + v_k \frac{\partial F_{ij}}{\partial x_k} = L_{ik} F_{kj}, \tag{2}$$

where t denotes time and the superposed dot denotes the material time derivative. In practice, (2) must be solved simultaneously with the momentum balance and constitutive law, and is subject to an initial condition that $\mathbf{F} = \mathbf{1}$ when the ice is first deposited at the surface. The strain $\mathbf{B}$, unit vectors $\mathbf{e}^{(r)}$ $(r = 1, 2, 3)$ and squares of the principal stretches b_r $(r = 1, 2, 3)$ are defined by

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T, \qquad \mathbf{B}\mathbf{e}^{(r)} = b_r \mathbf{e}^{(r)}, \qquad \det(\mathbf{B} - b_r \mathbf{1}) = 0. \tag{3}$$

The latter relation is a cubic with positive roots, and we adopt the ordering

$$b_1 \geqslant b_2 \geqslant b_3 > 0, \tag{4}$$

with strict inequalities except when the ice is in an undeformed isotropic state $\mathbf{B} = \mathbf{1}$, $b_1 = b_2 = b_3 = 1$. That is, we assume that the maximum compression is in the $\mathbf{e}_3$ direction.

By incompressibility,

$$\operatorname{div} \mathbf{v} = 0, \quad b_1 b_2 b_3 = 1, \quad b_1 > 1, \quad b_3 < 1, \tag{5}$$

but the sign of $(b_2 - 1)$ is not fixed. The structure tensors are defined by

$$\mathbf{M}^{(r)} = \mathbf{e}^{(r)} \otimes \mathbf{e}^{(r)}, \quad (r = 1, 2, 3). \tag{6}$$

The deviatoric Cauchy stress is defined in terms of the Cauchy stress $\boldsymbol{\sigma}$ and mean pressure p by

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + p\mathbf{1}, \quad p = -\tfrac{1}{3}\operatorname{tr}\boldsymbol{\sigma}, \quad \operatorname{tr}\hat{\boldsymbol{\sigma}} = 0, \tag{7}$$

where p is a workless constraint not given by a constitutive law, but determined by momentum balance and boundary conditions.

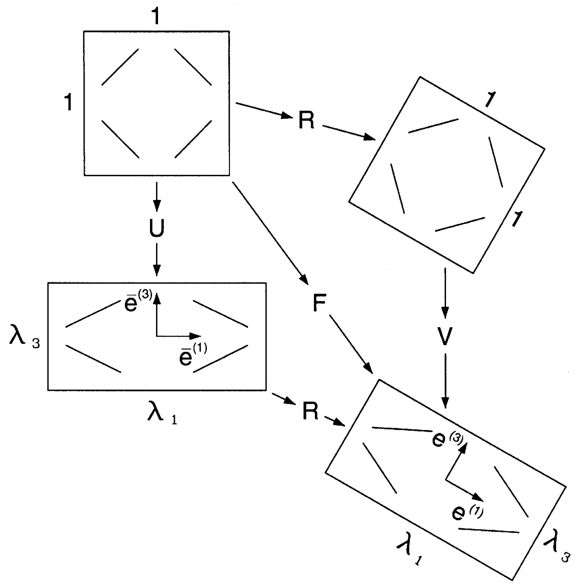


Figure 1. Plane view of deformation and rotation of a symmetric quadruple of glide planes.

Figure 1 illustrates a plane view of the deformation gradient tensor **F** of a polycrystal aggregate in terms of its two unique polar decompositions

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{8}$$

where **R** is the rotation tensor and **U**, **V** are, respectively, the positive definite right and left stretch tensors. The principal stretches are λ_r $(r = 1, 2, 3)$, along the principal axes $\bar{\mathbf{e}}^{(r)}$ (unit vectors) of **U** in the first decomposition, and along the principal axes $\mathbf{e}^{(r)}$ of **V** in the second decomposition, and

$$\bar{\mathbf{e}}^{(r)} = \mathbf{R}\mathbf{e}^{(r)}, \qquad b_r = \lambda_r{}^2, \quad (r = 1, 2, 3). \tag{9}$$

Newly formed compacted ice near the surface of an ice sheet is supposed macroscopically isotropic, due to the random distribution of individual crystal glide planes; that is, all glide planes and not just the basal planes. So, in the plane view in Figure 1, any crystal glide plane will have three others symmetrically oriented with respect to the chosen axes, illustrated here by the four basal planes of a set of symmetrically oriented crystals. As the aggregate deforms, these material planes are rotated towards a plane normal to a principal compression axis, $\lambda_r < 1$, and away from that normal to a principal extension axis, $\lambda_r > 1$. Except when $\lambda_1 = \lambda_2 = \lambda_3 = 1$, an undeformed state, all glide planes are rotated. Their symmetric distribution implies that reflexional symmetry in the three orthogonal principal stretch planes is maintained, either viewed in the non-rotated axes $\bar{\mathbf{e}}^{(r)}$, or rotated axes $\mathbf{e}^{(r)}$. Since the crystal basal glide planes are those planes over which the ice can shear most easily, this view implies that macroscopic shearing over the principal stretch planes should have ease of shearing, fluidities or reciprocal viscosities, ordered by the respective normal compressions, the inverse stretches λ_r^{-1}. Furthermore, the relative magnitudes of such viscosities should depend on the mean rotations and hence on, at least, the stretches λ_r^{-1}. An instantaneous viscous response must therefore include dependence on at least the principal stretches, as arguments of response coefficients, but possibly more generally on the deformation. The most simple approach to an instantaneous viscous constitutive law which captures an evolving orthotropic fabric suggested by the above picture is to relate the Cauchy deviatoric stress to the strain-rate, strain and the three structure tensors. All interactions between individual crystals which influence rotation are ignored in this relation.

The derivation of integrity (polynomial) and function bases for frame-indifferent (objective) relations between tensors and vectors, to ensure that material properties are independent of the observer, was pioneered by Rivlin and his associates (see, for example, Ericksen and Rivlin [13], Rivlin [15] and Smith and Rivlin [16], and reviews by Spencer [17], [18]). Here we are concerned only with a symmetric tensor relation for deviatoric stress in terms of strain-rate, strain and three structure tensors defining orthotropic response with respect to the current principal stretch planes, the form most convenient to use with the momentum equation. An alternative expression for strain-rate in terms of the other variables, the usual glaciology approach for the isotropic fluid model, can be formulated similarly. In order to include the commonly adopted viscous rate factor $a(T)$, where T denotes temperature, we introduce a modified strain-rate

$$\tilde{\mathbf{D}} = \frac{\mathbf{D}}{a(T)}. \tag{10}$$

The general orthotropic representation given by Boehler [19] is then

$$\begin{aligned} \hat{\boldsymbol{\sigma}} \; = \; & \sum_{r=1}^{3} [\phi_r \mathbf{M}^{(r)} + \phi_{r+3}(\mathbf{M}^{(r)}\tilde{\mathbf{D}} + \tilde{\mathbf{D}}\mathbf{M}^{(r)}) + \phi_{r+6}(\mathbf{M}^{(r)}\mathbf{B} + \mathbf{B}\mathbf{M}^{(r)})] \\ & + \phi_{10}\tilde{\mathbf{D}}^2 + \phi_{11}\mathbf{B}^2 + \phi_{12}(\tilde{\mathbf{D}}\mathbf{B} + \mathbf{B}\tilde{\mathbf{D}}), \end{aligned} \tag{11}$$

where the 12 response coefficients ϕ_i $(i = 1, \ldots, 12)$ are functions of the 19 invariants

$$\begin{aligned} & I_r = \operatorname{tr} \mathbf{M}^{(r)}\tilde{\mathbf{D}}, \qquad I_{r+3} = \operatorname{tr} \mathbf{M}^{(r)}\mathbf{B}, \qquad I_{r+6} = \operatorname{tr} \mathbf{M}^{(r)}\tilde{\mathbf{D}}^2, \\ & I_{r+9} = \operatorname{tr} \mathbf{M}^{(r)}\mathbf{B}^2, \qquad I_{r+12} = \operatorname{tr} \mathbf{M}^{(r)}\tilde{\mathbf{D}}\mathbf{B} \quad (r = 1, 2, 3), \\ & I_{16} = \operatorname{tr} \tilde{\mathbf{D}}^2\mathbf{B}, \qquad I_{17} = \operatorname{tr} \tilde{\mathbf{D}}\mathbf{B}^2, \qquad I_{18} = \det \tilde{\mathbf{D}}, \qquad I_{19} = \det \mathbf{B}, \end{aligned} \tag{12}$$

subject to the constraints that the deviatoric Cauchy stress has zero trace, and the material is incompressible, so that only 11 coefficients ϕ_i are independent, and only 18 invariants I_j are nontrivial; $I_{19} = 1$. While the above generality is beyond the restricted simpler models, we would expect to capture the main features of the ice response, and which could be correlated with observations, it is presented to demonstrate that the viscous properties of the response derived shortly are completely general, not a consequence of any particular restriction.

We also require that (11) reduces to an isotropic viscous fluid law

$$\hat{\boldsymbol{\sigma}} = \Phi_1\tilde{\mathbf{D}} + \Phi_2(\tilde{\mathbf{D}}^2 - \tfrac{1}{3}\operatorname{tr}\tilde{\mathbf{D}}^2\mathbf{1}), \tag{13}$$

where Φ_1, Φ_2 depend on the two invariants of $\tilde{\mathbf{D}}$, when there is no fabric; that is, in the initial undeformed state $\mathbf{F} = \mathbf{1}$ when the principal stretches are equal, necessarily $\lambda_1 = \lambda_2 = \lambda_3 = 1$ by incompressibility, or subsequently when $\mathbf{F} = \mathbf{1}$ or when $\mathbf{F} = \mathbf{R}$ which is a rigid rotation of the element. The conventional glaciology model is $\Phi_2 = 0$ and Φ_1 depends only on $\operatorname{tr}\tilde{\mathbf{D}}^2$.

The above prescription asserts that there is fabric – some alignment of initially random glide planes – only when there are differential principal stretches, or some shear, from the initial state. Now

$$\sum_{r=1}^{3} \mathbf{M}^{(r)} = \mathbf{1}, \tag{14}$$

so comparison of (13) and (11) implies that when $\mathbf{B} = \mathbf{1}$, and all orthogonal axes are principal stretch axes,

$$\begin{aligned} \mathbf{B} = \mathbf{1}: \quad & \phi_4 = \phi_5 = \phi_6\,, \qquad \phi_4 + \phi_{12} = \tfrac{1}{2}\Phi_1\,, \\ & \phi_1 = \phi_2 = \phi_3 = \tfrac{1}{3}\,\mathrm{tr}\,\tilde{\mathbf{D}}^2\ \Phi_2\,, \quad \phi_{10} = \Phi_2, \\ & \phi_7 = \phi_8 = \phi_9 = \phi_{11} = 0, \end{aligned} \tag{15}$$

and the invariants (12) become

$$\begin{aligned} \mathbf{B} = \mathbf{1}: \quad & I_r = \mathrm{tr}\,\mathbf{M}^{(r)}\tilde{\mathbf{D}}, \qquad I_{r+3} = \mathrm{tr}\,\mathbf{M}^{(r)} = 1, \\ & I_{r+6} = \mathrm{tr}\,\mathbf{M}^{(r)}\tilde{\mathbf{D}}^2, \qquad I_{r+9} = \mathrm{tr}\,\mathbf{M}^{(r)} = 1, \\ & I_{r+12} = I_r \quad (r = 1, 2, 3), \\ & I_{16} = \mathrm{tr}\,\tilde{\mathbf{D}}^2, \qquad I_{17} = \mathrm{tr}\,\tilde{\mathbf{D}} = 0, \qquad I_{18} = \det\tilde{\mathbf{D}}, \end{aligned} \tag{16}$$

implying dependence, at $\mathbf{B} = \mathbf{1}$, on the combinations

$$\begin{aligned} \mathbf{B} = \mathbf{1}: \quad & I_{21} = \textstyle\sum_{r=1}^{3} I_r = 0, \qquad I_{22} = \sum_{r=1}^{3} I_{r+6} = \mathrm{tr}\,\tilde{\mathbf{D}}^2, \\ & I_{23} = \textstyle\sum_{r=1}^{3} I_{r+12} = 0, \qquad I_{16} = \mathrm{tr}\,\tilde{\mathbf{D}}^2, \qquad I_{18} = \det\tilde{\mathbf{D}}. \end{aligned} \tag{17}$$

The restriction $\mathrm{tr}\,\hat{\boldsymbol{\sigma}} = 0$ provides one relation between the coefficients ϕ_i. Also $\mathrm{tr}\,\tilde{\mathbf{D}} = 0$ and $\det\mathbf{B} = 1$, and the invariant I_{19} is therefore not required. Since at any state we suppose a viscous response in which $\hat{\boldsymbol{\sigma}}$ vanishes when $\tilde{\mathbf{D}}$ vanishes, it is necessary that the coefficients ϕ_1, ϕ_2, ϕ_3, ϕ_7, ϕ_8, ϕ_9, ϕ_{11} vanish when $\tilde{\mathbf{D}}$ vanishes; that is, when I_1, I_2, I_3, I_7, I_8, I_9, I_{13}, I_{14}, I_{15}, I_{16}, I_{17}, I_{18} vanish.

3. Directional viscosities

Consider distinct axial stretches λ_1, λ_2, λ_3 along the fixed co-ordinate axes x_1, x_2, x_3, corresponding to a deformation

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3, \qquad \lambda_1\lambda_2\lambda_3 = 1,$$

$$\mathbf{V} = \mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad \mathbf{R} = \mathbf{1}, \qquad \mathbf{B} = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}, \tag{18}$$

where X_1, X_2, X_3 are particle co-ordinates in the initial isotropic reference state. The velocity and strain-rates are

$$v_1 = x_1\dot{\lambda}_1/\lambda_1, \qquad v_2 = x_2\dot{\lambda}_2/\lambda_2, \qquad v_3 = x_3\dot{\lambda}_3/\lambda_3,$$

$$\mathbf{D} = \begin{pmatrix} \dot{\lambda}_1/\lambda_1 & 0 & 0 \\ 0 & \dot{\lambda}_2/\lambda_2 & 0 \\ 0 & 0 & \dot{\lambda}_3/\lambda_3 \end{pmatrix}. \tag{19}$$

The principal stretch axes $\mathbf{e}^{(r)}$ and $\bar{\mathbf{e}}^{(r)}$ coincide with the co-ordinate axes, so

$$\mathbf{M}^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{M}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{M}^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{20}$$

and the deviatoric stress is given by the diagonal tensor

$$\hat{\boldsymbol{\sigma}} = \begin{pmatrix} \hat{\sigma}_1 & 0 & 0 \\ 0 & \hat{\sigma}_2 & 0 \\ 0 & 0 & \hat{\sigma}_3 \end{pmatrix}, \tag{21}$$

where the components are defined in terms of the principal stresses σ_1, σ_2, σ_3 by

$$\hat{\sigma}_1 = \tfrac{2}{3}\,\sigma_1 - \tfrac{1}{3}\,(\sigma_2 + \sigma_3), \qquad \hat{\sigma}_2 = \tfrac{2}{3}\,\sigma_2 - \tfrac{1}{3}\,(\sigma_1 + \sigma_3), \qquad \hat{\sigma}_3 = \tfrac{2}{3}\,\sigma_3 - \tfrac{1}{3}\,(\sigma_1 + \sigma_2). \tag{22}$$

The invariants (12) are functions of λ_i, $\dot{\lambda}_i$ $(i = 1, 2, 3)$.

Now remove the stress, and hence strain-rate, so the fabric defined by the current λ_1, λ_2, λ_3 is frozen, and consider the new instantaneous responses to simple shearings in different directions on different co-ordinate planes. For simple shear in the x_i direction on a plane normal to the x_j direction $(i \neq j)$, with no summation implied by a repeated suffix,

$$x_i = \lambda_i X_i + \kappa_{ij} X_j\,, \qquad x_j = \lambda_j X_j\,, \qquad x_k = \lambda_k X_k\,, \tag{23}$$

$$v_i = \dot{\kappa}_{ij} x_j/\lambda_j\,, \qquad v_j = v_k = 0, \qquad D_{ij} = \tfrac{1}{2}\,\dot{\kappa}_{ij}/\lambda_j\,, \tag{24}$$

where i, j, k are distinct permutations of 1, 2, 3, and the other strain-rate components are zero except the symmetric entries D_{ji}. Figure 2 illustrates the deformations for $i = 1$, $j = 3$ and $i = 3$, $j = 1$. Instantaneously, at the frozen values of λ_1, λ_2, λ_3, the tensors **F**, **R**, **B** are given by the diagonal tensors (18), and $\mathbf{M}^{(r)}$ $(r = 1, 2, 3)$ by the single diagonal elements

(20). The symmetric generators in (11) have instantaneous (ij) components, equal to the (ji) components,

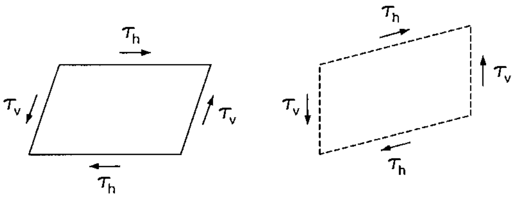


Figure 2. Simple shear deformation parallel and normal to the plane $x_3 = $ constant.

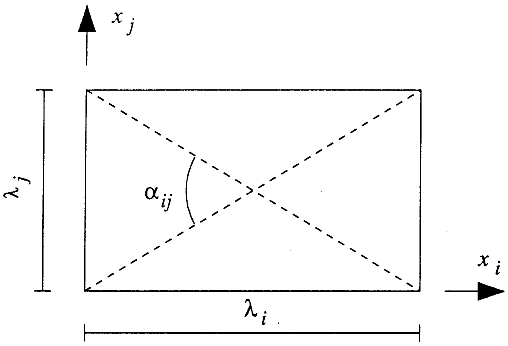


Figure 3. Principal stretches λ_i and λ_j in the principal stretch plane Ox_ix_j.

$$\mathbf{M}^{(r)}\mathbf{D} + \mathbf{D}\mathbf{M}^{(r)} : D_{ij} \quad (r = i \text{ or } j), \qquad 0 \quad (r \neq i \text{ or } j), \tag{25}$$

$$\mathbf{M}^{(r)}\mathbf{B} + \mathbf{B}\mathbf{M}^{(r)} : 0, \qquad \mathbf{D}^2 : 0, \qquad \mathbf{B}^2 : 0, \qquad \mathbf{DB} + \mathbf{BD} : (b_i + b_j)\, D_{ij}\,, \tag{26}$$

recalling (9), but there are also nonzero components other than (ij), so the instantaneous stress is not simply the shear components $\sigma_{ij} = \sigma_{ji}$.

The (ij) component $(i \neq j)$ of the constitutive relation (11) therefore has the instantaneous expression

$$\sigma_{ij} = [\phi_{i+3} + \phi_{j+3} + (b_i + b_j)\,\phi_{12}]\,\tilde{D}_{ij}, \tag{27}$$

defining an instantaneous viscosity for shear in the x_i direction on a plane normal to the x_j direction by

$$\mu_{ij} = \frac{\sigma_{ij}}{2\tilde{D}_{ij}} = \tfrac{1}{2}\,[\phi_{i+3} + \phi_{j+3} + (b_i + b_j)\,\phi_{12}], \tag{28}$$

which depends for each (ij) only on the response coefficients ϕ_{i+3}, ϕ_{j+3} and ϕ_{12}, independent of other terms in the general relation (11). In particular, note that the terms $\phi_{10}\,\tilde{\mathbf{D}}^2$ in (11) and $\Phi_2\,\tilde{\mathbf{D}}^2$ in (13) are not detected by this response. The ratios of the instantaneous directional viscosities, from (28), are

$$\frac{\mu_{13}}{\mu_{23}} = \frac{\phi_4 + \phi_6 + (b_1 + b_3)\phi_{12}}{\phi_5 + \phi_6 + (b_2 + b_3)\phi_{12}}, \qquad \frac{\mu_{12}}{\mu_{13}} = \frac{\phi_4 + \phi_5 + (b_1 + b_2)\phi_{12}}{\phi_4 + \phi_6 + (b_1 + b_3)\phi_{12}}. \tag{29}$$

If the values of b_1 and b_2 are interchanged in the first ratio, for any b_3, then that ratio must become μ_{23}/μ_{13} with the original values, and similarly interchanging the values of b_2 and b_3 for any b_1 in the second ratio. Thus ϕ_{12} must not change when b_1, b_2, b_3 are permuted, the values of ϕ_4 and ϕ_5 are interchanged when b_1 and b_2 are interchanged, the values of ϕ_5 and ϕ_6 are interchanged when b_2 and b_3 are interchanged, and those of ϕ_4 and ϕ_6 when b_1 and

b_3 are interchanged. That is, ϕ_{12} can depend in the frozen fabric only on the combinations of invariants

$$\phi_{12} : I_{24} = \sum_{r=1}^{3} I_{r+3} = \mathrm{tr}\,\mathbf{B}, \qquad I_{25} = \sum_{r=1}^{3} I_{r+9} = \mathrm{tr}\,\mathbf{B}^2, \tag{30}$$

while ϕ_4, ϕ_5 and ϕ_6 can have common dependence on I_{24} and I_{25} and common dependence on $I_4 = b_1$, $I_5 = b_2$ and $I_6 = b_3$, respectively.

We now derive equalities and inequalities between the directional viscosities, corresponding to different sets of b_1, b_2, b_3, not noted by Morland and Staroszczyk [14]. With the ordering (4) there are 6 distinct sets of relative values of b_1, b_2 and b_3. The basic concept of easy glide plane rotations being governed by the relative magnitudes of the principal stretches leads, for each set, to corresponding equalities or inequalities of the directional viscosities.

Consider principal stretches λ_i and λ_j in the principal stretch plane Ox_ix_j. Figure 3 illustrates the rotations of the diagonals of an initial unit square when $\lambda_j < \lambda_i$ and $\alpha_{ij} < \pi/2$, where $\tan(\alpha_{ij}/2) = \lambda_j/\lambda_i$. It is evident that each intersection line of any set of symmetric glide planes with Ox_ix_j undergoes rotation towards the Ox_i axis which increases as α_{ij} decreases; that is, there is increasing alignment of c-axes towards the direction of a smaller principal stretch as it decreases relative to the other stretches. Thus the fluidity μ_{ij}^{-1} increases as α_{ij} decreases, or equivalently the viscosity μ_{ij} increases as α_{ij} increases; that is, as λ_j/λ_i increases. With (4) and $(9)_2$, then,

$$\frac{b_3}{b_1} \leqslant \frac{b_3}{b_2} \Rightarrow \mu_{13} \leqslant \mu_{23}, \tag{31}$$

with the equality $\mu_{13} = \mu_{23}$ for $b_1 = b_2$, and similarly

$$\frac{b_3}{b_1} \leqslant \frac{b_2}{b_1} \Rightarrow \mu_{13} \leqslant \mu_{12}, \tag{32}$$

with the equality $\mu_{12} = \mu_{13}$ for $b_2 = b_3$. It follows from (31) and (32) that the minimum directional viscosity, bearing in mind the ordering (4), is always μ_{13}, irrespective of b_2. The relation between μ_{12} and μ_{23} is determined by the ratio

$$\left(\frac{b_2}{b_1}\right) \bigg/ \left(\frac{b_3}{b_2}\right) = \frac{b_2^2}{b_1 b_3} = b_2^3, \tag{33}$$

where the incompressibility condition $(5)_2$ has been used. Hence, depending on the magnitude of b_2 relative to unity, we have

$$b_2 > 1 : \mu_{12} > \mu_{23}, \qquad b_2 = 1 : \mu_{12} = \mu_{23}, \qquad b_2 < 1 : \mu_{12} < \mu_{23}. \tag{34}$$

Now, in view of (31), (32) and (34), the relations between the directional viscosities μ_{ij} for the six possible sets of b_1, b_2, b_3, are

$$b_1 = b_2 = b_3 = 1 : \mu_{ij} = \mu \quad (i, j = 1, 2, 3), \tag{35}$$

$$b_1 = b_2 > 1 > b_3 : 0 < \mu_{13} = \mu_{23} < \mu_{12}, \tag{36}$$

$$b_1 > b_2 > 1 > b_3 : 0 < \mu_{13} < \mu_{23} < \mu_{12}, \tag{37}$$

$$b_1 > b_2 = 1 > b_3 : 0 < \mu_{13} < \mu_{23} = \mu_{12}, \tag{38}$$

$$b_1 > 1 > b_2 > b_3 : 0 < \mu_{13} < \mu_{12} < \mu_{23}, \tag{39}$$

$$b_1 > 1 > b_2 = b_3 : 0 < \mu_{13} = \mu_{12} < \mu_{23}, \tag{40}$$

where μ defines the corresponding isotropic fluid viscosity.

4. Model construction

Following Morland and Staroszczyk [14] we consider only the terms in (11) which contribute to, and can therefore be detected by, the instantaneous directional viscosities (28), and investigate a model relation

$$\hat{\boldsymbol{\sigma}} = \sum_{r=1}^{3} \phi_{r+3}[\mathbf{M}^{(r)}\tilde{\mathbf{D}} + \tilde{\mathbf{D}}\mathbf{M}^{(r)} - \tfrac{2}{3}\operatorname{tr}(\mathbf{M}^{(r)}\tilde{\mathbf{D}})\mathbf{1}] + \phi_{12}[\tilde{\mathbf{D}}\mathbf{B} + \mathbf{B}\tilde{\mathbf{D}} - \tfrac{2}{3}\operatorname{tr}(\tilde{\mathbf{D}}\mathbf{B})\mathbf{1}], \tag{41}$$

where the ϕ_{r+3} and ϕ_{12} terms have been modified to recover zero trace, noting that the included scalar $\operatorname{tr}(\mathbf{M}^{(r)}\tilde{\mathbf{D}}) = I_r$, and the scalar $\operatorname{tr}(\tilde{\mathbf{D}}\mathbf{B})$ is the sum of I_{r+12}. We further assume a separable dependence which factors out invariants depending only on the deformation $\mathbf{B}$ and retains a common dependence on invariants involving the strain-rate $\tilde{\mathbf{D}}$; that is

$$\phi_{12} = \Phi_{12}(I_{16}, I_{22})\, g(I_{24}, I_{25}),$$

$$\phi_{r+3} = \Phi_{12}(I_{16}, I_{22})\, f(I_{r+3}, I_{24}, I_{25}), \qquad (r = 1, 2, 3). \tag{42}$$

The directional viscosity (28) becomes

$$\mu_{ij} = \tfrac{1}{2}\,\Phi_{12}(I_{16}, I_{22})[f(b_i, I_{24}, I_{25}) + f(b_j, I_{24}, I_{25}) + (b_i + b_j)\, g(I_{24}, I_{25})], \tag{43}$$

and since the response coefficients (42) must yield the isotropic fluid law (13) when $\mathbf{B} = \mathbf{1}$, so $I_{24} = I_{25} = 3$,

$$\Phi_{12}(\operatorname{tr}\tilde{\mathbf{D}}^2, \operatorname{tr}\tilde{\mathbf{D}}^2) = \tfrac{1}{2}\Phi_1(\operatorname{tr}\tilde{\mathbf{D}}^2, \det\tilde{\mathbf{D}}), \qquad f(1,3,3) + g(3,3) = 1, \qquad \Phi_2 = 0. \tag{44}$$

An appropriate combination of ϕ_{10} and ϕ_{11} terms in (11) would be needed for a nonzero Φ_2 term in (13). Morland and Staroszczyk [14] investigated and illustrated the case of no dependence on I_{24} and I_{25}, when $f = f(b)$ and $g = 0$, combined with a constant viscosity isotropic response, and showed that this very simple single fabric response function model could reflect some proposed features qualitatively, but fails to allow flexibility for different directional viscosities. The viscosity inequalities (36)–(40) had not been derived.

We now focus on a model with two fabric response functions, slightly more general than that illustrated in [14],

$$\Phi_1 = \Phi_1(\operatorname{tr}\tilde{\mathbf{D}}^2), \qquad \Phi_2 = 0, \qquad f = f(b), \qquad g = g(\operatorname{tr}\mathbf{B}), \qquad f(1) + g(3) = 1, \tag{45}$$

and show that the new viscosity equalities and inequalities can be satisfied. For (45) the instantaneous viscosity (43) simplifies to

$$\mu_{ij} = \frac{\sigma_{ij}}{2\tilde{D}_{ij}} = \frac{\Phi_1(\mathrm{tr}\,\tilde{\mathbf{D}}^2)}{4}\,[f(b_i) + f(b_j) + (b_i + b_j)\,g(\mathrm{tr}\,\mathbf{B})], \tag{46}$$

and since this must remain bounded as any axial stretch increases indefinitely, we rewrite g and the normalisation $(45)_5$ as

$$g(K) = K^{-1}G(K), \quad K = \mathrm{tr}\,\mathbf{B} = b_1 + b_2 + b_3, \qquad f(1) + \tfrac{1}{3}G(3) = 1, \tag{47}$$

where $G(K)$ is bounded. Then the viscosity (43) becomes

$$\mu_{ij} = \frac{\mu}{2}[f(b_i) + f(b_j) + (b_i + b_j)\,K^{-1}G(K)], \quad \mu = \frac{\Phi_1(\mathrm{tr}\,\tilde{\mathbf{D}}^2)}{2}, \tag{48}$$

where μ defines the isotropic fluid viscosity at temperature T when $b_1 = b_2 = b_3 = 1$. By incompressibility (5) and the ordering (4),

$$1 \geqslant b_3 = 1/(b_1 b_2), \qquad K = b_1 + b_2 + 1/(b_1 b_2) \geqslant 3. \tag{49}$$

The equality (35) follows from the normalisation $(45)_5$. Define

$$h_{12} = f(b_1) - f(b_3) + \frac{b_1 - b_3}{K}G(K) \propto (\mu_{12} - \mu_{23}), \tag{50}$$

$$h_{21} = f(b_2) - f(b_1) + \frac{b_2 - b_1}{K}G(K) \propto (\mu_{23} - \mu_{13}), \tag{51}$$

$$h_{23} = f(b_2) - f(b_3) + \frac{b_2 - b_3}{K}G(K) \propto (\mu_{12} - \mu_{13}), \tag{52}$$

then (35)–(40) become

$$b_1 = b_2 > 1 > b_3 : h_{21} = 0, \quad h_{12} > 0, \tag{53}$$

$$b_1 > b_2 > 1 > b_3 : h_{21} > 0, \quad h_{12} > 0, \tag{54}$$

$$b_1 > b_2 = 1 > b_3 : h_{21} > 0, \quad h_{12} = 0, \tag{55}$$

$$b_1 > 1 > b_2 > b_3 : h_{23} > 0, \quad h_{12} < 0, \tag{56}$$

$$b_1 > 1 > b_2 = b_3 : h_{23} = 0, \quad h_{12} < 0. \tag{57}$$

The equalities $h_{21} = 0$ in (53) and $h_{23} = 0$ in (57) are automatically satisfied, while the equality $h_{12} = 0$ in (55), when $b_2 = 1$, hence $b_3 = 1/b_1$, gives a relation for $G(K)$ in terms of $f(b)$; namely

$$G(K) = -\frac{K\,b_1}{b_1^2 - 1}[f(b_1) - f(b_1^{-1})], \quad K \geqslant 3, \tag{58}$$

where

$$2\,b_1 = K - 1 + \sqrt{(K-1)^2 - 4}, \quad \geqslant 2. \tag{59}$$

The limit of (58) as $b_1 \to 1$, $K \to 3$, combined with the normalisation $(45)_5$, shows that

$$G(3) = -3\,f'(1) = 3\,[1 - f(1)], \tag{60}$$

which is a restriction on $f(b)$ at $b = 1$. The limit as $b_1 \to \infty$, $K \sim b_1$, yields

$$G(\infty) = f(0) - f(\infty). \tag{61}$$

That is, only one fabric response function $f(b)$ remains free for prescription, subject to (60), to match with observed properties, but $g(K)$ and the coefficient ϕ_{12} cannot vanish as supposed in Morland and Staroszczyk [14]. They also proposed, and satisfied, the inequality $\mu_{12} < \mu_{13}$ when $b_2 > b_3$, which violates the above deductions. The remaining inequalities of (53)–(57) require that

$$h_{21} > 0, \quad h_{12} \gtrless 0 \quad \text{for} \;\; b_2 \gtrless 1, \quad h_{23} > 0 \quad \text{for} \;\; b_2 < 1. \tag{62}$$

It is reasonable to expect that $f(b)$ and $G(K)$ are monotonic functions and do not change sign, then, by (58) and (60),

$$f'(b) \gtrless 0 \Leftrightarrow G(K) \lessgtr 0, \qquad f(1) - 1 \gtrless 0 \Leftrightarrow G(3) \lessgtr 0. \tag{63}$$

We will see that two further experimental limit relations, coupled with (61), determine a negative $G(\infty)$, so then (63) implies that $f(b)$ is monotonic increasing. A variety of simple increasing functions $f(b)$ will be proposed to demonstrate numerically that the inequalities (62) can be achieved, and to illustrate continued stretching and shearing responses.

5. Enhancement factors

Budd and Jacka [20] and Li Jun *et al.* [21] determine experimentally the limit ratios of fabric induced viscosity to isotropic viscosity for indefinite axial compression with equal unconfined lateral extensions, and for indefinite shear in a plane deformation following compression and stretching. The reciprocals of these ratios are described as enhancement factors. We can now determine these ratios when the constitutive model (41) with the fabric functions (45) is adopted.

In the first experiment there are equal lateral stretches $\lambda_1 = \lambda_2 > 1$, and by incompressibility the axial stretch (a compression) is $\lambda_3 = \lambda_1^{-2} < 1$. The deformation is described by (18)–(20) with the above identities, for which

$$\operatorname{tr} \tilde{\mathbf{D}}\mathbf{B} = 2\tilde{D}_{11}(b_1 - b_1^{-2}), \qquad K = 2b_1 + b_1^{-2}. \tag{64}$$

The law (41) with (45) yields

$$\frac{\hat{\sigma}_{11}}{2\mu\tilde{D}_{11}} = \frac{\hat{\sigma}_{33}}{2\,\mu\,\tilde{D}_{33}} = \tfrac{1}{3} f(b_1) + \tfrac{2}{3} f(b_1^{-2}) + \frac{G(K)}{3K}(b_1 + 2b_1^{-2}). \tag{65}$$

As $b_1 \to \infty$, with $K \sim 2b_1$,

$$\frac{\hat{\sigma}_{11}}{2\mu\tilde{D}_{11}} = \frac{\hat{\sigma}_{33}}{2\mu\tilde{D}_{33}} \to \tfrac{1}{3}f(\infty) + \tfrac{2}{3}f(0) + \tfrac{1}{6}G(\infty), = A, \tag{66}$$

where A is the reciprocal of the axial enhancement factor.

Next consider an initial plane compression and stretch which is frozen at constant $\lambda_3 = \lambda_1^{-1}$ by the removal of the stress and strain-rate, then followed by a simple shear at constant strain-rate $D_{13} = \frac{1}{2}\dot{\gamma}$ defined by

$$x_1 = \lambda_1 X_1 + \kappa X_3, \qquad x_2 = X_2, \qquad x_3 = \lambda_1^{-1} X_3, \quad \dot{\kappa} = \dot{\gamma}\lambda_1^{-1}, \tag{67}$$

$$\mathbf{B} = \begin{pmatrix} \lambda_1^2 + \kappa^2 & 0 & \lambda_1^{-1}\kappa \\ 0 & 1 & 0 \\ \lambda_1^{-1}\kappa & 0 & \lambda_1^{-2} \end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix} 0 & 0 & \frac{1}{2}\dot{\gamma} \\ 0 & 0 & 0 \\ \frac{1}{2}\dot{\gamma} & 0 & 0 \end{pmatrix}. \tag{68}$$

The principal stretch squares b_i $(i = 1, 2, 3)$, the eigenvalues of $\mathbf{B}$, are

$$b_2 = 1, \qquad b_3 = b_1^{-1}, \qquad 2b_1 = \lambda_1^2 + \lambda_1^{-2} + \kappa^2 + \sqrt{\left(\lambda_1^2 + \lambda_1^{-2} + \kappa^2\right)^2 - 4}, \tag{69}$$

and the associated principal vectors $\mathbf{e}^{(r)}$ are given by

$$\begin{aligned} &\mathbf{e}^{(2)} = (0, 1, 0), \qquad e_2^{(s)} = 0, \qquad \lambda_1^{-1}\kappa\, e_1^{(s)} + (\lambda_1^{-2} - b_s)\, e_3^{(s)} = 0, \\ &[e_1^{(s)}]^2 + [e_3^{(s)}]^2 = 1 \quad (s = 1, 3). \end{aligned} \tag{70}$$

Thus the structure tensors and required combinations and invariants are

$$\mathbf{M}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{M}^{(s)} = \begin{pmatrix} e_1^{(s)}e_1^{(s)} & 0 & e_1^{(s)}e_3^{(s)} \\ 0 & 0 & 0 \\ e^{(s)} - 1e_3^{(s)} & 0 & e_3^{(s)}e_3^{(s)} \end{pmatrix} \quad (s = 1, 3), \tag{71}$$

$$\begin{aligned} &\mathbf{M}^{(2)}\mathbf{D} + \mathbf{D}\mathbf{M}^{(2)} = \mathbf{0}, \\ &\mathbf{M}^{(s)}\mathbf{D} + \mathbf{D}\mathbf{M}^{(s)} = \tfrac{1}{2}\dot{\gamma} \begin{pmatrix} 2e_1^{(s)}e_3^{(s)} & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2e_1^{(s)}e_3^{(s)} \end{pmatrix} \quad (s = 1, 3), \end{aligned} \tag{72}$$

$$\operatorname{tr}\mathbf{M}^{(2)}\mathbf{D} = 0, \qquad \operatorname{tr}\mathbf{M}^{(s)}\mathbf{D} = \dot{\gamma}\, e_1^{(s)}e_3^{(s)} \quad (s = 1, 3). \tag{73}$$

The ϕ_{12} term requires

$$\mathbf{DB} + \mathbf{BD} = \tfrac{1}{2}\dot{\gamma} \begin{pmatrix} 2\lambda_1^{-1}\kappa & 0 & \lambda_1^2 + \lambda_1^{-2} + \kappa^2 \\ 0 & 0 & 0 \\ \lambda_1^2 + \lambda_1^{-2} + \kappa^2 & 0 & 2\lambda_1^{1}\kappa \end{pmatrix}, \quad \operatorname{tr}\mathbf{DB} = \dot{\gamma}\lambda_1^{-1}\kappa. \tag{74}$$

By (41) and (45),

$$\frac{\hat{\sigma}_{11}}{\mu\dot{\gamma}} = \frac{\hat{\sigma}_{33}}{\mu\dot{\gamma}} = -\frac{\hat{\sigma}_{22}}{2\mu\dot{\gamma}} = \frac{1}{3}\left[f(b_1)\, e_1^{(1)} e_3^{(1)} + f(b_1^{-1})\, e_1^{(3)} e_3^{(3)} + \frac{G(K)}{K}\lambda_1^{-1}\kappa \right], \tag{75}$$

$$\frac{\hat{\sigma}_{13}}{\mu\dot{\gamma}} = \frac{\hat{\sigma}_{31}}{\mu\dot{\gamma}} = \frac{1}{2}\left[f\big(b_1\big) + f(b_1^{-1}) + \frac{G(K)}{K}(\lambda_1^2 + \lambda_1^{-2} + \kappa^2) \right], \tag{76}$$

where $K = b_1 + 1 + b_1^{-1}$. As $\kappa \to \infty$ with λ_1 finite, $b_1 \sim \kappa^2$, $K \sim \kappa^2$, so (76) implies that

$$\frac{\hat{\sigma}_{13}}{\mu\dot{\gamma}} \to \tfrac{1}{2} f(\infty) + \tfrac{1}{2} f(0) + \tfrac{1}{2} G(\infty), = S, \tag{77}$$

where S is the reciprocal of the shear enhancement factor. The three linear relations (61), (66) and (77) for $f(\infty)$, $f(0)$ and $G(\infty)$ have a unique solution

$$f(0) = S, \qquad f(\infty) = 6A - 5S, \qquad G(\infty) = 6(S - A). \tag{78}$$

For illustration we choose the values

$$S = \tfrac{1}{8}, \qquad A = \tfrac{1}{3}, \tag{79}$$

measured by Budd and Jacka [20] for warm ice near melting, but recognise that their test conditions are not necessarily appropriate to the response of ice in a cold sheet. In particular, it is expected that for cold ice A exceeds unity (an enhancement factor for axial compression less than unity – according to Pimienta *et al.* [22] it can be less than 0·1). The limit values (78) are then

$$f(0) = \tfrac{1}{8}, \qquad f(\infty) = \tfrac{11}{8}, \qquad G(\infty) = -\tfrac{5}{4}, \tag{80}$$

which supports the earlier suggestion from (63) that $G(K)$ is negative and $f(b)$ is monotonic increasing, with $f(1) > 1$, and further that $f(b)$ is positive. There are the additional restrictions (60) on f at $b = 1$ and G at $K = 3$. For any choice of $f(b)$ meeting these conditions it remains to satisfy the inequalities (62).

6. Model validity and illustrations

We now explore some simple monotonic increasing functions $f(b)$ with free parameters, determine numerically the corresponding $G(K)$, and demonstrate that the required inequalities can be satisfied. For illustration purposes, two following fabric response functions $f(b)$ have been adopted

$$f(b) = f_\infty - (f_\infty - f_0)\, \exp(-\alpha b^n), \quad \alpha > 0, \quad n > 0, \tag{81}$$

$$f(b) = f_0 + (f_\infty - f_0)\, \tanh(\alpha b), \quad \alpha > 0, \tag{82}$$

where $f_0 = f(0)$ and $f_\infty = f(\infty)$ are limit values prescribed by (78)–(80), n in (81) is a free parameter, and α in both (81) and (82) is determined by the restriction $(60)_2$. Plots of the chosen fabric functions $f(b)$ are presented in Figure 4, where the curves labelled (1), (2),

(3) correspond to the function (81) with $n = 1,\ 2,\ 3$ respectively, and the curve labelled (4) corresponds to the function (82). The same labelling applies in subsequent illustrations. Figure 5 shows plots of the functions $G(K)$ determined by (58) for this set of functions $f(b)$. We note that the function (81) with $n \gtrsim 2$ yields non-monotonic functions $G(K)$, with strong minimums for large n. It is expected that the latter is an undesirable feature which will lead to an unlikely viscous response for the ice, and this will be confirmed.

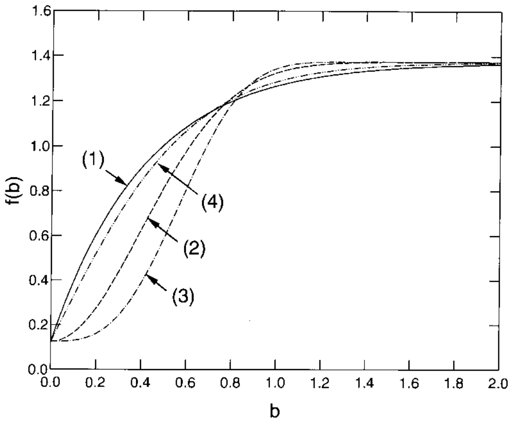


Figure 4. Adopted forms of the fabric response function $f(b)$.

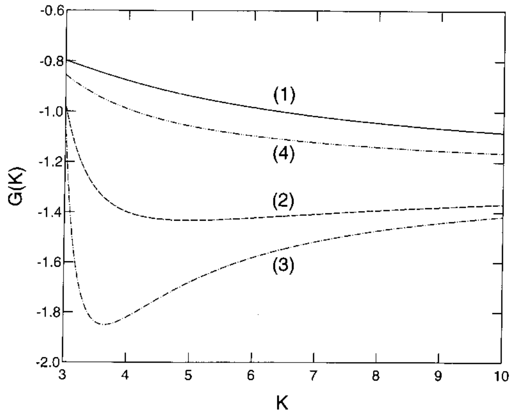


Figure 5. Functions $G(K)$ associated with the adopted fabric response functions $f(b)$.

With the adopted response functions presented in Figures 4 and 5, numerical simulations of the uniaxial unconfined compression and simple shearing tests, as described in Section 5, have been carried out. The results for the uniaxial compression started from the initially isotropic state are shown in Figure 6, in which the evolution of the normalised axial viscosity $\hat{\sigma}_{33}/(2\mu\tilde{D}_{33})$ with increasing λ_1 is illustrated for the different functions $f(b)$. It is seen that the function (81) with $n = 1$ and the function (82) yield very similar results, while (81) with $n = 2$ predicts much faster softening of the ice, and (81) with $n \gtrsim 3$ gives rise to non-monotonic response which is an unexpected and unlikely material response. For $n \gtrsim 5$ there are physically invalid responses with negative viscosity.

The results of simulations of the simple shearing following the compression along the x_3-axis in a plane flow $b_2 = 1$ are shown in Figure 7, in which the evolution of the normalised directional viscosity $\hat{\sigma}_{13}/(\mu\dot{\gamma})$ with increasing shear κ is presented. Figure 7(a) illustrates the results obtained for different functions $f(b)$ in the case of shearing started from the isotropic state $\lambda_1 = \lambda_3 = 1$, and Figure 7(b) presents the results obtained for shearing started from an anisotropic state induced by an initial compression $\lambda_3 = 1/\lambda_1 = 0{\cdot}5$. Again, the functions (81) with $n = 1$ and (82) give similar results, and (81) with $n \gtrsim 3$ again yields an unlikely response.

Finally, Figure 8 shows, for the function (81) with $n = 1$, the evolution of dimensionless directional viscosities μ_{ij}/μ with b_1 for all (except the isotropic state) possible cases of relative values $b_1 \geqslant b_2 \geqslant b_3$. It is seen from the plots that all the equalities and inequalities (36)–(40) are satisfied for this particular choice of the fabric response function $f(b)$, and

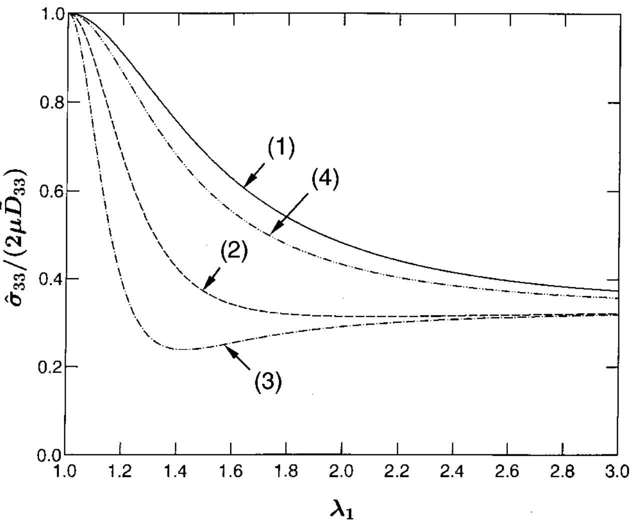
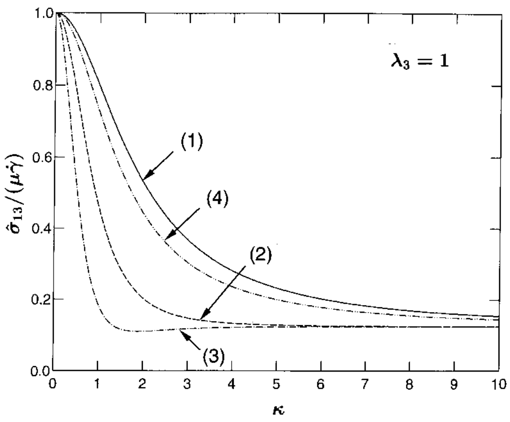


Figure 6. Evolution of the ratio $\hat{\sigma}_{33}/(2\mu\tilde{D}_{33})$ with λ_1 in uniaxial compression for different fabric response functions $f(b)$.



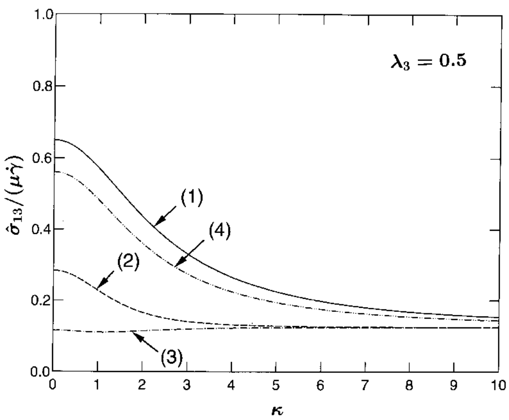


Figure 7. Evolution of the ratio $\hat{\sigma}_{13}/(\mu\dot{\gamma})$ with increasing shear κ following initial plane compression λ_3 for different fabric response functions $f(b)$: (a) shearing starts from an isotropic state ($\lambda_3 = 1$); (b) shearing starts from an anisotropic state ($\lambda_3 = 0{\cdot}5$).

monotonic decreasing viscosities are obtained for all the cases considered. The function (82) yields very similar results. However, while the inequalities are verified for the other values of n in the function (81), it has been seen that non-monotonic responses occur for $n \gtrsim 3$, and negative viscosities for $n \gtrsim 5$.

7. Conclusions

An orthotropic viscous relation with reflexional symmetries in an evolving set of principal stretch planes successfully models observed features of fabric evolution in polar ice sheets. While the motivation for the macroscopic law is an underlying concept of individual crys-

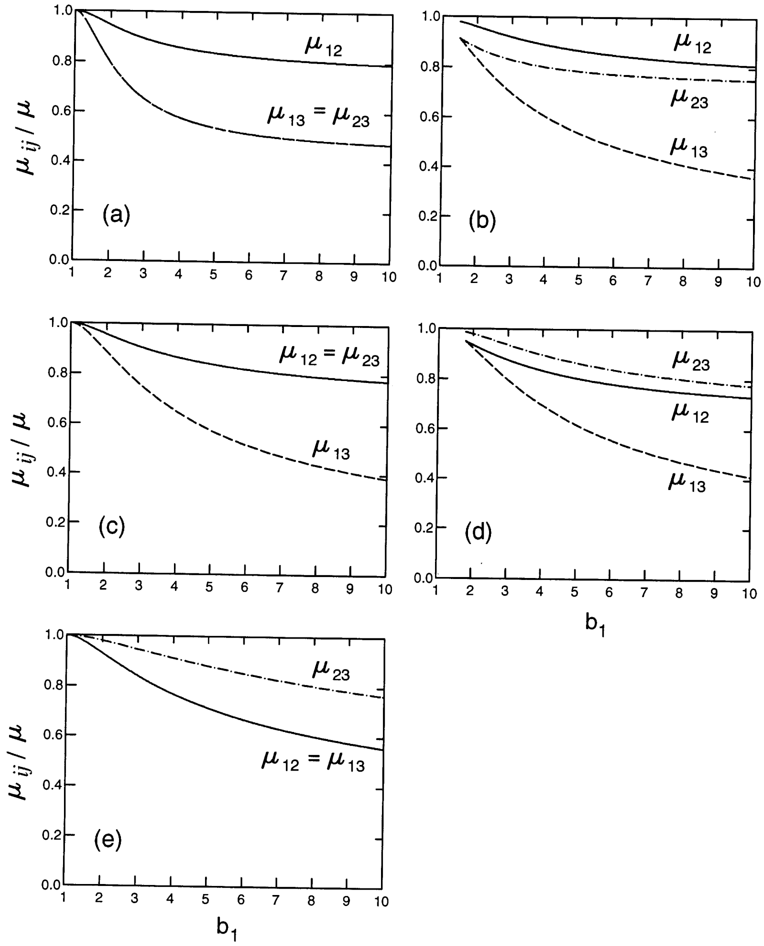


Figure 8. Evolution of dimensionless directional viscosities μ_{ij}/μ for different flows: (a) $b_1 = b_2 > 1 > b_3$ (uniaxial compression); (b) $b_1 > b_2 > 1 > b_3$ ($b_2 = 1{\cdot}5$); (c) $b_1 > b_2 = 1 > b_3$ (plane flow); (d) $b_1 > 1 > b_2 > b_3$ ($b_2 = 0{\cdot}75$); (e) $b_1 > 1 > b_2 = b_3$ (uniaxial extension).

tal glide plane rotations, such a law could reflect induced anisotropy in other materials for which the current response is instantaneously viscous. Illustrations show that very simple restricted forms can satisfy derived properties of directional viscosities and match measured enhancement factors in indefinite stretch and shear tests. There is ample flexibility to correlate response with further data, for ice or other materials, and this constitutive form provides a framework which could incorporate the macroscopic effects of a variety of observed crystal interaction processes.

Acknowledgements

This research was supported by an EU contract *Fabric Development and Rheology of Polar Anisotropic Ice for Ice Sheet Flow Modelling.*

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